

### Elementary Matrices

In this section, we will combine what we have learned so far into one coherent theory. We will carefully explain how to solve systems of equations via matrix operations rather than row operations. Given a system of equation written in matrix form  $A\mathbf{x} = \mathbf{b}$ , we can multiply both sides by several matrices, called *elementary matrices*, to obtain an equivalent system that is in row-echelon form. We will also be able to use these elementary matrices to compute the inverse of nonsingular matrices.

Given an  $m \times n$  system of equations  $A\mathbf{x} = \mathbf{b}$ , we can obtain an equivalent system by multiplying both sides by any nonsingular matrix  $M$ , like so:

$$MA\mathbf{x} = M\mathbf{b}.$$

If this is done more than once, in other words if we multiply by more than one nonsingular matrix (since they will be elementary matrices, we'll denote them by  $E_1, E_2, \dots, E_k$ ), we will obtain something like:

$$E_k \dots E_2 E_1 A\mathbf{x} = E_k \dots E_2 E_1 \mathbf{b}.$$

(Note that the matrices were multiplied on the left and that they were done in order.) This new system will be equivalent to the original provided that  $E_k \dots E_2 E_1$  is nonsingular. The following theorem will help in this regard.

**Theorem 1:** If  $A$  and  $B$  are nonsingular  $n \times n$  matrices, then  $AB$  is also a nonsingular  $n \times n$  matrix and  $(AB)^{-1} = B^{-1}A^{-1}$ .

**Proof:**

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$
$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

By using induction, it can be seen that this theorem can be extended to any finite product of nonsingular matrices. This theorem is only helpful if elementary matrices are actually nonsingular. They are, and we will see this as soon as we define the elementary matrices.

**Definition:** If we start with the identity matrix and perform exactly one row operation on it, the resulting matrix is called an *elementary matrix*. Since there are three row operations, there are three types of elementary matrices. These will be illustrated in the following three examples.

**Example 1:** (Type I) Define  $E_s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . This is the matrix that results from

switching rows 1 and 2 of the identity matrix. Multiplying a matrix by a matrix of this type will interchange the same two rows. For example,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

**Example 2:** (Type II) Define  $E_m = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . This is the matrix that results from

multiplying row 1 of the identity matrix by 3. Multiplying a matrix by a matrix of this type will do the same. For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 3 \\ -3 & 4 & 4 \\ 1 & 2 & 1 \end{bmatrix}.$$

**Example 3:** (Type III) Define  $E_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ . This is the matrix that results from

adding 4 times row 1 to row 3 in the identity matrix. Multiplying a matrix by a matrix of this type will do the same. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 4 & 4 \\ 9 & 2 & 5 \end{bmatrix}.$$

**Theorem 2:** If  $E$  is an elementary matrix, then  $E$  is nonsingular and  $E^{-1}$  is an elementary matrix of the same type.

**Proof:** Multiplying an elementary matrix of type I by itself clearly produces the identity matrix. So every type I elementary matrix is nonsingular and is its own inverse. A similar argument shows that type II and III matrices are nonsingular. In these cases though, the elementary matrices are not their own inverses, but the inverse matrices are still of the same type.

**Theorem 3:** Let  $A$  be an  $n \times n$  matrix. Then the following are equivalent:

- (a)  $A$  is nonsingular.
- (b)  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution  $\mathbf{x} = \mathbf{0}$ .
- (c)  $A$  is row equivalent to the identity matrix.

A theorem of this type (“the following are equivalent”) is especially powerful. What it says is that if any one of those three conditions is true, then so are the other two. It also implies the following result.

**Corollary:** The system  $A\mathbf{x} = \mathbf{b}$  has a unique solution if and only if  $A$  is nonsingular.

Suppose we have a nonsingular matrix  $A$  and we wish to find its inverse. By Theorem 3,  $A$  is row equivalent to the identity matrix. So there exist elementary matrices  $E_1, E_2, \dots, E_k$  such that  $E_k \dots E_2 E_1 A = I$ . Multiplying both sides on the right by  $A^{-1}$  yields  $E_k \dots E_2 E_1 I = A^{-1}$ . Thus the same series of row operations that transform a nonsingular matrix into the identity will transform the identity matrix into  $A^{-1}$ . So if we augment the matrix  $A$  with the identity matrix (which is denoted by  $[A|I]$ ) and put the left into reduced row echelon form, the right will turn into  $A^{-1}$ . This is demonstrated in the next example.

**Example 4:** Find the inverse of the nonsingular matrix  $A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$ .

The augmented matrix is  $\left[ \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array} \right]$ . Putting the left side into reduced row echelon form produces:

$$\left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & -2 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & 1 \end{array} \right]$$

$$\left[ \begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{7} & -\frac{2}{7} \end{array} \right]$$

$$\left[ \begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{1}{7} \\ 0 & 1 & \frac{3}{7} & -\frac{2}{7} \end{array} \right]$$

$$\text{So } A^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{2}{7} \end{bmatrix}.$$

## Homework

1. For each of the following pairs of matrices, find an elementary matrix  $A$  such that  $A$ .

$$(a) A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$$

$$(b) A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix}$$

2. Let  $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$ .

(a) Verify that  $A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$ .

(b) Use  $A^{-1}$  to solve  $A\mathbf{x} = \mathbf{b}$  for each of the following choices for  $\mathbf{b}$ .

(i)  $\mathbf{b} = [1 \ 1 \ 1]^T$       (ii)  $\mathbf{b} = [1 \ 2 \ 3]^T$       (iii)  $\mathbf{b} = [-2 \ 1 \ 0]^T$

3. Find the inverse of each of the following matrices.

(a)  $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$       (b)  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$       (c)  $A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

4. Let  $A$  and  $B$  be  $n \times n$  matrices and let  $C = AB$ . Prove that if  $B$  is singular, then  $C$  must be singular also. [Hint: Use Theorem 3.]